The Kozai Mechanism and the Evolution of Binary Supermassive Black Holes

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ABSTRACT

We consider the dynamical evolution of bound, hierarchical triples of supermassive black holes that might be formed in the nuclei of galaxies undergoing sequential mergers. The tidal force of the outer black hole on the inner binary produces eccentricity oscillations through the Kozai mechanism, and this can substantially reduce the gravitational wave merger time of the inner binary. We numerically calculate the merger time for a wide range of initial conditions and black hole mass ratios, including the effects of octupole interactions in the triple as well as general relativistic periastron precession in the inner binary. The semimajor axes and the mutual inclination of the inner and outer binaries are the most important factors affecting the merger time. We find that for a random distribution of inclination angles and approximately equal mass black holes, it is possible to reduce the merger time of a near circular inner binary by more than a factor of ten in over fifty percent of all cases. We estimate that a typical exterior quadrupole moment from surrounding matter in the galaxy may also be sufficient to excite eccentricity oscillations in supermassive black hole binaries, and also accelerate black hole mergers.

1. INTRODUCTION

The ubiquity of supermassive black holes (SMBHs) in the nuclei of many galaxies (e.g. Magorrian et al. 1998) suggests that binary and multiple SMBH systems may also be widespread. For example, SMBH binaries should form in galaxy mergers (Begelman, Blandford, & Rees 1980), which is a common process in hierarchical models for galaxy formation and evolution (e.g. Kauffmann & Haehnelt 2000; Menou, Haiman, & Narayanan 2001). SMBH binaries have been invoked to explain periodic wiggles in extragalactic radio jets (e.g. Roos, Kaastra, & Hummel 1993), periodic flares in the BL Lac source OJ 287 (e.g. Lehto & Valtonen 1996), variations in the apparent superluminal transverse velocities and position angles of the 3C 273 radio jet (Romero et al. 2000), and the "core-type" nuclear surface brightness profiles of bright elliptical galaxies (Ebisuzaki, Makino, & Okumura 1991; Quinlan & Hernquist 1997; Faber et al. 1997; Milosavljević & Merritt 2001).

SMBH binary mergers would be powerful sources of gravitational waves (Thorne & Braginsky 1976) which should be easily detectable from planned space-based gravitational wave observatories such as the *Laser Interferometer Space Antenna* (Bender et al. 1998). However, it is still uncertain

whether binary SMBHs can evolve to small enough semimajor axes that gravitational radiation drives them to merge. Energy can be extracted from a wide binary through interactions with surrounding matter. For example, encounters with passing stars will shrink the binary, but this process will eventually be limited by the depletion of stars on sufficiently radial orbits to encounter the ever tightening binary. Detailed work on this aspect of the problem has been done by numerous authors (e.g. Quinlan 1996, Quinlan & Hernquist 1997, Milosavljević & Merritt 2001). A key uncertainty is the amount of stochastic wandering that the binary undergoes at the bottom of the potential well of the galaxy, allowing it to interact with many more stars. Another important process that has received relatively less attention is interaction with surrounding gas, which can also extract energy from the binary and drive it to merge (e.g. Armitage & Natarajan 2002 and references therein).

Large galaxies are typically the product of multiple mergers over the history of the universe. If the characteristic merger time of binary SMBHs is not much less than the characteristic time scale between mergers, then interactions with a third SMBH or a second SMBH binary will likely take place. Strong encounters within these multiple black hole systems may lead to slingshot ejections of SMBHs from the nucleus of the galaxy (Valtonen et al. 1994, Valtonen 1996). On the other hand, the very existence of the additional galaxy merger may introduce more stellar encounters with the original central binary and drive it to merge (Roos 1988). Moreover, repeated encounters between a third black hole and the inner binary can increase the probability that the inner binary attains a large eccentricity, thereby accelerating the energy and angular momentum loss by gravitational radiation (Makino & Ebisuzaki 1994).

A modification of the last scenario is a case where the third black hole has evolved to the point that it has become bound to the SMBH binary, but has not yet come close enough for an unstable three-body interaction to take place. The system then forms a hierarchical triple and can be treated as consisting of an inner binary and an outer binary. In that case, if the mutual inclination angle between the inner and outer binaries is high enough, then the time-averaged tidal gravitational force on the inner binary can induce an oscillation in its eccentricity. This effect is known as the Kozai mechanism (Kozai 1962). Provided all other dynamical influences are negligible (see section 2), an initially very small eccentricity in the inner binary will oscillate through a maximum value given by

$$e_{1,\max} \simeq \left(1 - \frac{5}{3}\cos^2 i_0\right)^{1/2}, \tag{1}$$

provided $|\cos i_0| < (3/5)^{1/2}$, i.e. the initial mutual inclination angle i_0 lies between roughly 39° and 141° (Kozai 1962). Given that the gravitational wave merger time is a strong function of eccentricity (approximately proportional to $[1 - e_1^2]^{7/2}$), highly inclined orbits could in principle greatly reduce the merger time. For example, a mutual inclination angle of 56° results in eccentricity oscillations which, when at maximum amplitude, reduce the characteristic merger time by an order of magnitude. Although the true merger time will depend on how long the inner binary spends at high eccentricity, it appears promising that a random distribution of initial inclinations will result

in dramatically accelerated mergers in many cases. This is the subject we will explore in the present paper.

While this research was being completed, we learned of recent work by Miller & Hamilton (2002), who investigate a very similar idea: the use of the Kozai mechanism to accelerate binary mergers in triple black hole systems formed in globular clusters.

We begin in section 2 by going through the characteristic time scales which will affect the dynamics of triple supermassive black hole systems. Then in section 3 we summarize the evolution equations of an isolated triple used in our numerical calculations. Section 4 presents the results of those calculations, which help delineate the regions of initial condition parameter space where the Kozai mechanism plays a substantial role in accelerating the merger of the inner binary. We discuss our results and summarize our conclusions in section 5.

2. CHARACTERISTIC TIME SCALES

Consider a binary system consisting of two black holes with masses m_0 and m_1, with initial semimajor axis a_1 and eccentricity e_1. The time it takes for the binary to merge due to gravitational wave emission (Peters 1964) can be written as

$$\begin{aligned} t_{\mathrm{merge,binary}} \simeq\ & 2.9 \times 10^{12}\mathrm{yr} \left(\frac{m_0}{10^6\,\mathrm{M}_\odot}\right)^{-1} \left(\frac{m_1}{10^6\,\mathrm{M}_\odot}\right)^{-1} \left(\frac{m_0+m_1}{2\times 10^6\,\mathrm{M}_\odot}\right)^{-1} \left(\frac{a_1}{10^{-2}\mathrm{pc}}\right)^{4} \\ & \times f(e_1)(1-e_1^2)^{7/2}, \end{aligned} \tag{2}$$

where $f(e_1)$ is a weak function of the initial eccentricity that is of order unity ($0.979 < f(e_1) < 1.81$ for all e_1). This gravitational wave merger time is a strong function of both the semimajor axis and the eccentricity. Hence gravitational radiation only becomes important late in the evolution of the binary.

Prior to that time, the evolution is dominated by interactions between the binary and surrounding material, either stars or gas. These interactions are very complex, due to the fact that the surroundings themselves evolve due to their interaction with the binary. (See Milosavljević & Merritt 2001 for a recent discussion of stellar interactions.) As we noted above, it is not yet clear whether these interactions are sufficient to harden the binary to a point where gravitational radiation then causes it to merge.

The scenario we wish to explore in this paper is one where the binary's semimajor axis evolution has stalled because of insufficient interactions with surrounding material, and where a subsequent galaxy merger then introduces a third black hole in the system. We envisage the third black hole eventually forming a bound, hierarchical triple with the binary. The tidal gravitational torques exerted on the inner binary by the outer black hole can alter the eccentricity of the inner binary, thereby affecting the rate of gravitational wave emission.

General relativity causes periastron precession in the inner binary, with a period

$$P_{\rm GR} \simeq 2.3 \times 10^6 {\rm yr} \left(\frac{m_0+m_1}{2\times 10^6\,{\rm M}_\odot}\right)^{-3/2} \left(\frac{a_1}{10^{-2}{\rm pc}}\right)^{5/2} (1-e_1^2). \tag{3}$$

Provided this general relativistic precession is unimportant, and the outer black hole is in a sufficiently inclined orbit around the inner binary, then the eccentricity of the inner binary will oscillate by the Kozai mechanism. The characteristic time scale for these oscillations is given by (e.g. Holman, Touma, & Tremaine 1997)

$$P_e \simeq 1.3 \times 10^5 {\rm yr} \left(\frac{m_0+m_1}{2\times 10^6\,{\rm M}_\odot}\right)^{-1/2} \left(\frac{a_1}{10^{-2}{\rm pc}}\right)^{3/2} \left(\frac{m_0+m_1}{2m_2}\right) \left(\frac{a_2/a_1}{10}\right)^{3} (1-e_2^2)^{3/2}, \tag{4}$$

where m_2 is the mass of the third body and a_2 and e_2 are the semimajor axis and eccentricity, respectively, of its orbit around the inner binary.

General relativistic precession can stop these eccentricity oscillations by destroying the Kozai resonance (e.g. Holman et al. 1997). A precise criterion for this not to happen is easily derived (see equation [A6] in Appendix):[1]

$$\frac{a_2}{a_1} < 34 \left(\frac{a_1}{10^{-2}{\rm pc}}\right)^{1/3} \left(\frac{m_0+m_1}{2\times 10^6\,{\rm M}_\odot}\right)^{-1/3} \left(\frac{2m_2}{m_0+m_1}\right)^{1/3} \left(\frac{1-e_1^2}{1-e_2^2}\right)^{1/2}. \tag{5}$$

The outer black hole must therefore come quite close to the inner binary for eccentricity oscillations to take place, although the triple is still hierarchical in the sense that the semimajor axis ratio a_2/a_1 is large.

Interactions with the surrounding stars and gas may cause the outer black hole's semimajor axis to evolve significantly over the merger time scale of the inner binary, a point to which we shall return in section 5 below. For now, however, we will neglect this fact and consider the evolution of an isolated, hierarchical black hole triple. As we will show, once the outer black hole is sufficiently close that equation (5) is satisfied, the merger time of the inner binary drops substantially for sufficiently high mutual inclinations.

3. EQUATIONS OF MOTION OF AN ISOLATED TRIPLE SYSTEM

We assume that the triple is hierarchical, so that the orbit of the outer black hole is much larger in size than the orbit of the inner binary. In this case the triple can be considered to consist of two binaries in approximately Keplerian orbits: the inner binary consisting of black holes with masses m_0 and m_1; and the outer binary consisting of the center of mass of the inner binary, with

[1] Apart from the dependence on e_1 and factors of order unity, this is roughly equivalent to $P_e < P_{\rm GR}$. Equation (5) provides a more quantitatively accurate description of the behavior seen in our numerical results.

mass $m_0 + m_1$, and the third black hole with mass m_2. Let the semimajor axes of the inner and outer binaries be a_1 and a_2, respectively, and the eccentricities be e_1 and e_2. We also define g_1 and g_2 to be the corresponding arguments of periastron.

The magnitude of the angular momenta $\mathbf{G}_1$ and $\mathbf{G}_2$ of the inner and outer binaries are given by

$$G_1 = m_0 m_1 \left[\frac{G a_1 (1 - e_1^2)}{m_0 + m_1} \right]^{1/2} \tag{6}$$

and

$$G_2 = (m_0 + m_1) m_2 \left[\frac{G a_2 (1 - e_2^2)}{m_0 + m_1 + m_2} \right]^{1/2} \tag{7}$$

respectively, where G is Newton's gravitational constant. Let the total orbital angular momentum of the triple be $\mathbf{H} = \mathbf{G}_1 + \mathbf{G}_2$. In the absence of gravitational radiation, tidal torques, or gravitational interactions with surrounding stars, this vector would be rigorously conserved. Let i_1 (i_2) be the inclination of the inner (outer) binary, i.e. the angle between $\mathbf{G}_1$ ($\mathbf{G}_2$) and $\mathbf{H}$. Then

$$H = G_1 \cos i_1 + G_2 \cos i_2 \tag{8}$$

and

$$G_1 \sin i_1 = G_2 \sin i_2. \tag{9}$$

The mutual inclination angle between the two binaries is $i = i_1 + i_2$.

Marchal (1990), Krymolowski & Mazeh (1999) and Ford, Kozinsky, & Rasio (2000) have derived the orbit-averaged Hamiltonian of an isolated, Newtonian hierarchical triple of point masses, using secular perturbation theory to octupole order, i.e. to order $(a_1/a_2)^3$. We adopt the equations of motion for the orbital elements of Ford et al. here,[2] but modify them to incorporate two general relativistic effects on the inner binary: the precession of periastron and gravitational radiation. We do this by simply adding orbit-averaged general relativistic correction terms to the Ford et al. (2000) equations of motion for the orbital elements of the inner binary, the same correction terms which would exist if the inner binary were isolated. We must also take into account the fact that $\mathbf{H}$ is no longer conserved because of the radiative loss $(dG_1/dt)_{\rm rad}$ of the inner binary's orbital angular momentum. We do this by noting that in the absence of interactions between the inner and outer binaries, gravitational radiation acts to change the magnitude of $\mathbf{G}_1$, but not its direction. In addition, the vector $\mathbf{G}_2$ remains unchanged. The resulting equation is

$$\frac{dH}{dt} = \frac{G_1 + G_2 \cos i}{H} \left(\frac{dG_1}{dt} \right)_{\rm rad}. \tag{10}$$

[2]The equations of motion of Krymolowski & Mazeh (1999) differ from those of Ford et al. (2000) by terms of order $(a_1/a_2)^{7/2}$ resulting from the canonical transformation of the von Zeipel method. We have corrected a sign error in the octupolar terms in the equations of motion of Ford et al. (2000). In equations (22) and (29)-(32) of their paper, all terms involving C_3 should have the opposite sign. The equations of motion of Krymolowski & Mazeh (1999) also contain the same sign error.

Note that if the black holes are spinning, then the orbital plane of the inner binary can also evolve because of general relativistic spin-orbit coupling. This is a higher order effect than general relativistic precession, and we neglect it entirely, assuming that we are essentially dealing with Schwarzschild black holes.

Thus our evolution equations for the orbital elements are

$$\frac{da_1}{dt} = -\frac{64G^3 m_0 m_1 (m_0+m_1)}{5c^5 a_1^3 (1-e_1^2)^{7/2}} \left(1 + \frac{73}{24}e_1^2 + \frac{37}{96}e_1^4\right), \tag{11}$$

$$\begin{aligned}
\frac{dg_1}{dt} &= 6C_2 \left\{ \frac{1}{G_1}[4\theta^2 + (5\cos 2g_1 - 1)(1 - e_1^2 - \theta^2)] + \frac{\theta}{G_2}[2 + e_1^2(3 - 5\cos 2g_1)] \right\} \\
&\quad + C_3 e_2 e_1 \left(\frac{1}{G_2} + \frac{\theta}{G_1}\right) \{\sin g_1 \sin g_2 [A + 10(3\theta^2 - 1)(1 - e_1^2)] - 5\theta B \cos\phi\} \\
&\quad - C_3 e_2 \frac{1 - e_1^2}{e_1 G_1} \left[10\theta(1-\theta^2)(1 - 3e_1^2)\sin g_1 \sin g_2 + \cos\phi(3A - 10\theta^2 + 2)\right] \\
&\quad + \frac{3}{c^2 a_1 (1 - e_1^2)} \left[\frac{G(m_0+m_1)}{a_1}\right]^{3/2},
\end{aligned} \tag{12}$$

$$\begin{aligned}
\frac{de_1}{dt} &= 30C_2 \frac{e_1(1-e_1^2)}{G_1}(1-\theta^2)\sin 2g_1 \\
&\quad - C_3 e_2 \frac{1-e_1^2}{G_1}[35\cos\phi(1-\theta^2)e_1^2 \sin 2g_1 - 10\theta(1-e_1^2)(1-\theta^2)\cos g_1 \sin g_2 \\
&\quad - A(\sin g_1 \cos g_2 - \theta \cos g_1 \sin g_2)] \\
&\quad - \frac{304 G^3 m_0 m_1 (m_0+m_1) e_1}{15 c^5 a_1^4 (1-e_1^2)^{5/2}} \left(1 + \frac{121}{304}e_1^2\right),
\end{aligned} \tag{13}$$

$$\frac{da_2}{dt} = 0, \tag{14}$$

$$\begin{aligned}
\frac{dg_2}{dt} &= 3C_2 \left\{ \frac{2\theta}{G_1}[2 + e_1^2(3 - 5\cos 2g_1)] + \frac{1}{G_2}[4 + 6e_1^2 + (5\theta^2 - 3)(2 + 3e_1^2 - 5e_1^2\cos 2g_1)] \right\} \\
&\quad - C_3 e_1 \sin g_1 \sin g_2 \left\{ \frac{4e_2^2 + 1}{e_2 G_2} 10\theta(1-\theta^2)(1-e_1^2) \right. \\
&\qquad \left. - e_2 \left(\frac{1}{G_1} + \frac{\theta}{G_2}\right)[A + 10(3\theta^2 - 1)(1 - e_1^2)] \right\} \\
&\quad - C_3 e_1 \cos\phi \left[5B\theta e_2 \left(\frac{1}{G_1} + \frac{\theta}{G_2}\right) + \frac{4e_2^2 + 1}{e_2 G_2} A\right],
\end{aligned} \tag{15}$$

$$\frac{de_2}{dt} = C_3 e_1 \frac{1 - e_2^2}{G_2}\left[10\theta(1-\theta^2)(1-e_1^2)\sin g_1 \cos g_2 + A(\cos g_1 \sin g_2 - \theta \sin g_1 \cos g_2)\right], \tag{16}$$

and

$$\frac{dH}{dt} = -\frac{32G^3 m_0^2 m_1^2}{5c^5 a_1^3 (1-e_1^2)^2} \left[\frac{G(m_0+m_1)}{a_1}\right]^{1/2} \left(1+\frac{7}{8}e_1^2\right) \frac{G_1+G_2\theta}{H}. \tag{17}$$

The quantities C_2 and C_3 multiply the quadrupole and octupole perturbation terms, respectively. They are defined by (Ford et al. 2000)

$$C_2 = \frac{G m_0 m_1 m_2}{16(m_0+m_1)a_2(1-e_2^2)^{3/2}} \left(\frac{a_1}{a_2}\right)^2, \tag{18}$$

and

$$C_3 = \frac{15 G m_0 m_1 m_2 (m_0-m_1)}{64(m_0+m_1)^2 a_2(1-e_2^2)^{5/2}} \left(\frac{a_1}{a_2}\right)^3. \tag{19}$$

Note that the octupole terms vanish if $m_0 = m_1$. The quantities B and A in these terms are given by

$$B = 2 + 5e_1^2 - 7e_1^2 \cos 2g_1 \tag{20}$$

and

$$A = 4 + 3e_1^2 - \frac{5}{2}(1-\theta^2)B. \tag{21}$$

Finally, the quantity θ is the cosine of the mutual inclination of the binaries,

$$\theta = \cos i = \frac{H^2 - G_1^2 - G_2^2}{2G_1G_2}, \tag{22}$$

and ϕ is the angle between the periastron directions,

$$\cos\phi = -\cos g_1 \cos g_2 - \theta \sin g_1 \sin g_2. \tag{23}$$

Note that equation (22) determines the evolution of the mutual inclination through the time dependence of the eccentricities, a_1, and H. The terms in the equations of motion which remain after setting $C_2 = C_3 = 0$ are the general relativistic correction terms. The last term in equation (12) is the general relativistic precession term, and the terms involving the speed of light c in equations (11), (13), and (17) are the gravitational radiation terms.

We may immediately deduce some important features of our evolution equations. Equations (13) and (16) imply that if both the inner and outer binaries are circular ($e_1 = e_2 = 0$), then they will remain that way. An eccentric outer binary will produce a nonzero eccentricity in an initially circular inner binary, and vice-versa, but only because of the octupole interaction terms.

Note that if we switch off the quadrupole and octupole interaction terms by setting $C_2 = C_3 = 0$, then an immediate consequence is $d\theta/dt = 0$, as may be verified by differentiating equation (22) directly. In other words, general relativistic effects alone do not change the mutual inclination of the inner and outer binaries if there are no interactions between them. This is as one would expect, given our treatment of the gravitational wave angular momentum loss in equation (10).

Our evolution equations also exhibit an important scaling. If all masses and all initial semimajor axes are multiplied by some constant factor, then the merger time and all other time scales change by the same multiplicative factor. This is a direct consequence of the nature of gravity, where mass, length, and time have the same dimensions in units where $G = c = 1$, and it reduces the parameter space of masses and semimajor axes which need to be explored numerically. Another similarly useful fact is that the equations are unchanged when m_0 and m_1 in the inner binary are interchanged and g_1 or g_2 is increased by $180°$. The former corresponds to a relabeling of the inner binary black holes, while the latter represents a spatial inversion of the two orbits.

While our implementation of the general relativistic effects has been heuristic, the general relativistic precession terms for the inner binary can in fact be rigorously justified by orbit-averaging the post-Newtonian Hamiltonian. To octupole order, the resulting Hamiltonian is

$$\begin{aligned}\bar{\mathcal{H}}(g_1, g_2, G_1, G_2) \quad = \quad & C_2\left[(2+3e_1^2)(1-3\theta^2) - 15e_1^2(1-\theta^2)\cos 2g_1\right] \\ & + C_3 e_1 e_2 \left[A\cos\phi + 10\theta(1-\theta^2)(1-e_1^2)\sin g_1 \sin g_2\right] \\ & + \frac{G^2 m_0 m_1}{c^2 a_1^2}\left[\frac{15m_1^2 + 15m_0^2 + 29m_0 m_1}{8(m_0+m_1)} - \frac{3(m_0+m_1)}{(1-e_1^2)^{1/2}}\right]. \end{aligned} \tag{24}$$

A rigorous derivation of the radiation terms would be considerably more difficult, and is beyond the scope of this paper.[3] We believe that our equations of motion will capture the overall time evolution of the inner binary, because the early phases of this evolution will be dominated by interactions with the outer black hole, while the late phases will be dominated by gravitational radiation. In both those limits our evolution equations for the orbital elements of the inner binary are rigorously correct. It is possible, however, that interesting effects may occur during the transition between these two phases of evolution, which may not be captured by our equations.

In the absence of gravitational radiation, our equations conserve the orbit averaged Hamiltonian in equation (24). We use this fact to check the accuracy of our numerical integrations.

The reader should bear in mind that our equations are based on orbit averaging and an expansion in the semimajor axis ratio a_1/a_2, and may therefore be inaccurate when this ratio is not very small. We will show in the next section that the octupole terms usually have fairly small effects on the merger time of the inner binary. Hence our neglect of even higher order terms, and our use of such an expansion in the first place, is probably justified for the primary purpose of this paper.

The major exception to this is the issue of stability. We are investigating whether the presence of a third black hole can cause the inner binary to merge before the third black hole comes close enough to cause an unstable interaction. There is a limit to how small a_2/a_1 can be for the triple

[3]One of the issues is that the Newtonian portion of the equations of motion were derived by using the invariable plane as the reference plane. Because this plane slowly evolves under the influence of gravitational radiation, it may be that a rigorous set of equations of motion should include terms to reflect this slow evolution.

to be stable and for the evolution we calculate to be valid. Completely general stability criteria for hierarchical triples in arbitrary initial configurations do not exist. We adopt as a stability criterion the empirical formula used by Mardling & Aarseth (2001), which may be written as

$$\frac{a_2}{a_1} > \frac{2.8}{1-e_2}\left[\left(1+\frac{m_2}{m_0+m_1}\right)\frac{1+e_2}{(1-e_2)^{1/2}}\right]^{2/5}. \tag{25}$$

This criterion was derived for Newtonian coplanar prograde orbits of the inner and outer binaries. Inclined orbits, which are the major focus of this paper, are expected to be more stable, so inequality (25) provides a conservative stability limit. This inequality is also conservative in the sense that it is either close to or above the more complicated stability criterion proposed by Eggleton & Kiseleva (1995), which was empirically verified over a wide range of mass ratios including nearly all those we numerically investigate in the next section. All the calculations we present in this paper are done for systems which are stable according to inequality (25).

The stability criterion (25) may be combined with the constraint of inequality (5) that general relativistic precession not destroy the Kozai resonance to give a lower limit on the inner semimajor axis for which the Kozai mechanism can reduce the inner binary merger time. [4] The result is

$$a_1 \gtrsim 6\times 10^{-6}\mathrm{pc}\left(\frac{m_0+m_1+m_2}{m_0+m_1}\right)^{6/5}\left(\frac{m_0+m_1}{2m_2}\right)\left(\frac{m_0+m_1}{2\times 10^6\mathrm{M}_\odot}\right)\frac{(1+e_2)^{27/10}}{(1-e_2)^{21/10}(1-e_1^2)^{3/2}}. \tag{26}$$

A stable triple with initial conditions such that the inner semimajor axis a_1 violates this inequality will not have an accelerated inner binary merger by the Kozai mechanism. However, in that case a_1 is so small that the binary merger time by equation (2) is

$$\begin{aligned} t_{\mathrm{merge,binary}} \lesssim\ & 0.3\mathrm{yr}\left(\frac{m_0}{10^6\mathrm{M}_\odot}\right)^{-1}\left(\frac{m_1}{10^6\mathrm{M}_\odot}\right)^{-1}\left(\frac{m_0+m_1}{2\times 10^6\mathrm{M}_\odot}\right)^{3}\left(\frac{m_0+m_1+m_2}{m_0+m_1}\right)^{24/5} \\ & \left(\frac{m_0+m_1}{2m_2}\right)^{4}\frac{(1+e_2)^{54/5}f(e_1)}{(1-e_2)^{42/5}(1-e_1^2)^{5/2}}. \end{aligned} \tag{27}$$

At least for roughly equal mass triples and outer eccentricities that are not too high, this implies that the inner binary would already be rapidly merging and needs no help from the Kozai mechanism.

On the other hand, equations (26) and (27) have a rather strong dependence on e_2, so if the outer black hole's orbit is *very* eccentric, then the resulting triple may be too unstable for there to be time for the Kozai mechanism to operate. However, numerical simulations of black hole binary evolution by Milosavljević & Merritt (2001) suggest that this is rather unlikely. They find rather modest initial eccentricities when the binary first forms, and the subsequent evolution of the binary does not go to very high eccentricity. Hence there is probably plenty of parameter space for stable hierarchical triples to form and evolve under the Kozai mechanism.

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[4] We are grateful to the referee for suggesting this connection between stability and the precession constraint, and for emphasizing to us the importance of discussing the stability of triple systems in general.

4. NUMERICAL RESULTS

4.1. Detailed Evolution of a Triple Consisting of Nearly Equal Mass Black Holes

We have numerically integrated equations (11)-(17) for a wide range of possible black hole masses and initial conditions. Even for fixed black hole masses, the parameter space of possible initial conditions is huge, and it helps to understand which are the most important quantities affecting the merger time. We therefore focus first on the detailed evolution of a particular triple consisting of nearly equal mass black holes. Specifically, we consider an inner binary with a 2×10^6 $M_\odot$ black hole and a 10^6 $M_\odot$ black hole, about which orbits another 10^6 $M_\odot$ black hole ten times further out. We have purposely chosen the two inner black holes to have different masses so as to allow for octupole interaction effects in the evolution.

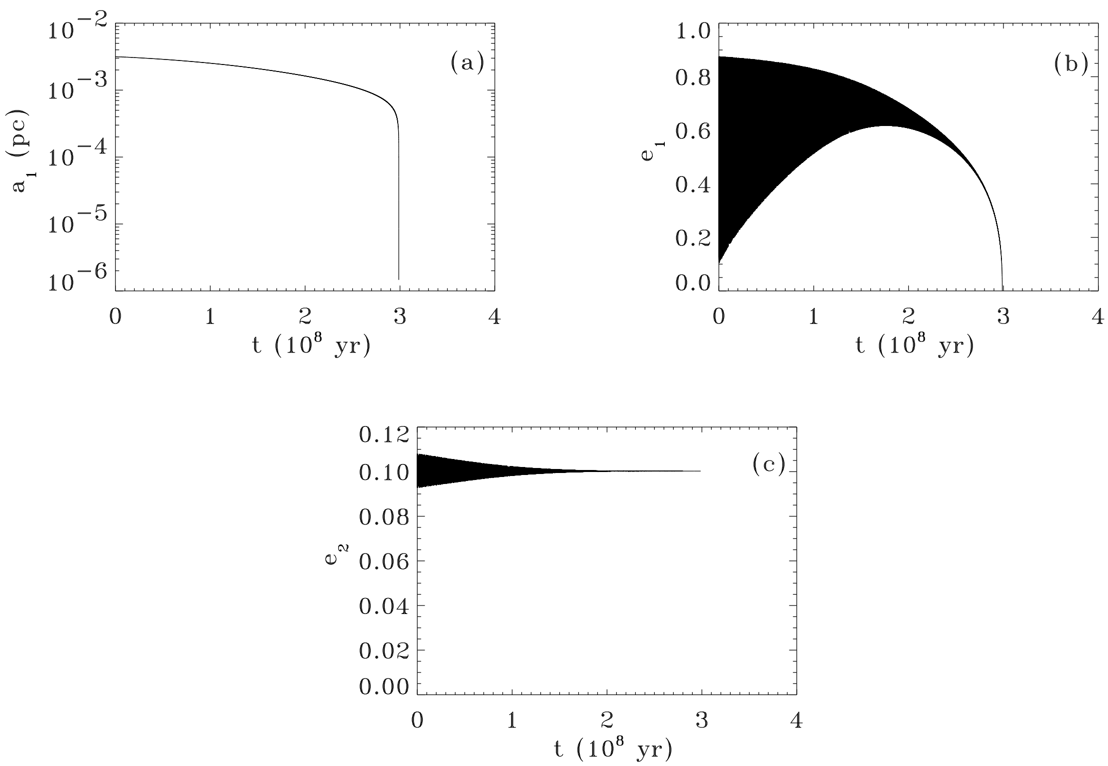

Fig. 1.— Evolution of (a) the inner binary semimajor axis a_1, (b) the inner binary eccentricity e_1, and (c) the outer binary eccentricity e_2 with time for a triple consisting of black hole masses $m_0 = 2 \times 10^6$ $M_\odot$ and $m_1 = m_2 = 10^6$ $M_\odot$. The initial conditions of the triple are $a_1 = 3.16 \times 10^{-3}$ pc, $a_2/a_1 = 10$, $e_1 = 0.1$, $e_2 = 0.1$, $g_1 = 0$, $g_2 = 90°$, and $i = 80°$. The large amplitude inner binary eccentricity oscillations due to the Kozai mechanism are clearly evident. This greatly accelerates the merger of the inner binary: in the absence of eccentricity oscillations, the binary would take 9.3×10^9 yr to merge.

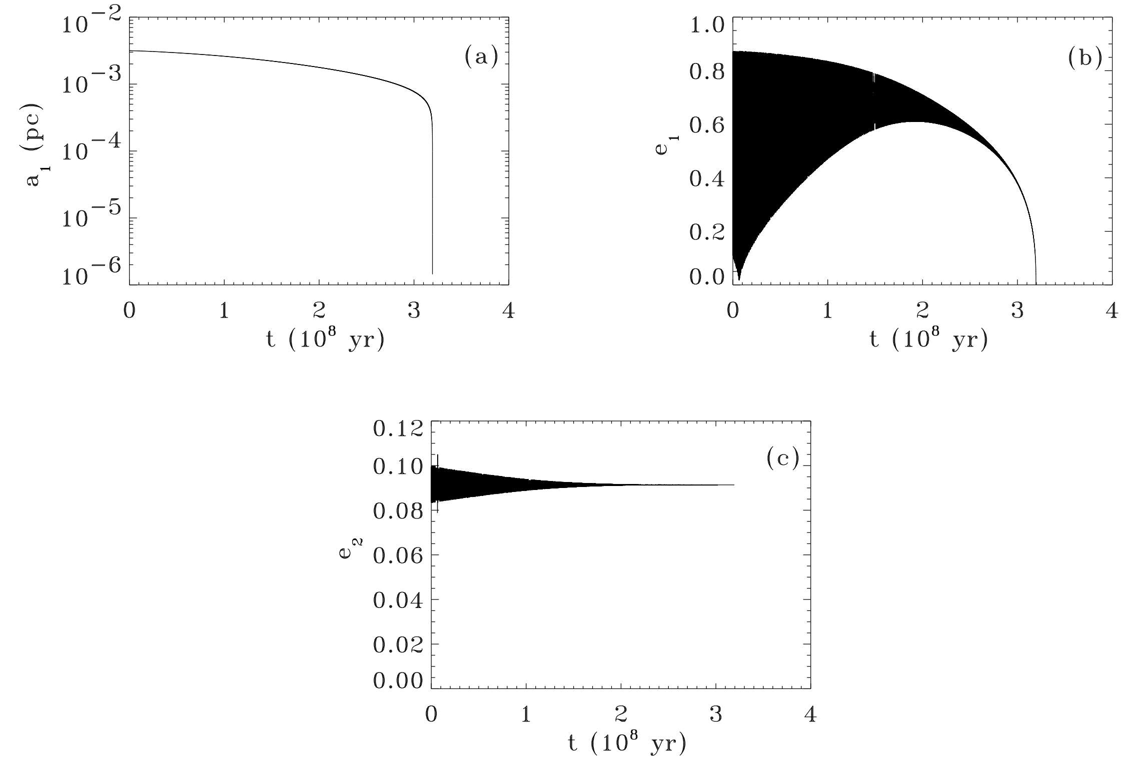

Fig. 2.— Same as figure 1 except for an initial inner argument of periastron $g_1 = 90°$.

Figure 1 depicts the evolution of inner semimajor axis a_1, inner eccentricity e_1, and outer eccentricity e_2 for a particular choice of initial conditions. The inner binary starts out nearly circular ($e_1 = 0.1$) with a semimajor axis which would give a gravitational wave merger time $t_{\rm merge,binary} = 9.3 \times 10^9$ yr if the binary were isolated. However, the presence of the outer black hole induces large amplitude eccentricity oscillations in the inner binary through the Kozai mechanism. The time spent at higher eccentricity reduces the gravitational wave merger time of the two inner black holes by roughly a factor thirty for the particular case shown.

In contrast to e_1, the outer eccentricity e_2 stays close to its original value throughout the evolution. Equation (16) shows that e_2 only changes as a result of octupole interactions, which are weaker than the quadrupolar interactions driving the inner binary eccentricity oscillations. The lack of strong evolution in the outer eccentricity is a generic feature of all the numerical calculations we have done, implying that e_2 is unlikely to be driven to high enough values to destabilize an isolated triple.

As might be expected, the merger time turns out to be rather insensitive to the initial orientation of the roughly circular inner orbit, i.e. the initial value of the inner argument of periastron g_1. Figure 2 shows the evolution of the same triple as depicted in figure 1, except that the initial

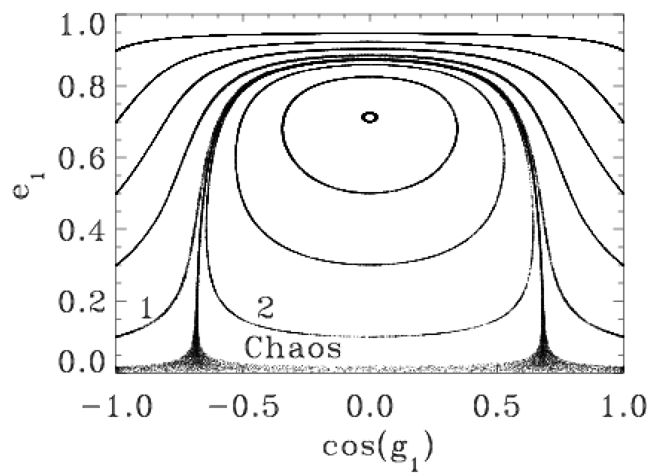

Fig. 3.— Trajectories in the e_1 vs. $\cos g_1$ phase space for triples with similar initial conditions to the triples in figures 1 and 2, neglecting gravitational radiation. Each curve corresponds to a different initial inner eccentricity e_1 and mutual inclination angle i chosen to keep the total angular momentum of the triple fixed. The initial inner argument of periastron g_1 is also chosen to be either 0 or 90°. Provided the initial e_1 is not too small, the former choice always produces circulation in g_1, while the latter can produce libration in g_1 about 90° instead. The circulating curve labeled 1 and the librating curve labeled 2 have identical initial conditions to the triples shown in figures 1 and 2, respectively. Triples which start out with nearly circular inner binaries (e_1 small) are chaotic, repeatedly crossing the separatrix between circulation and libration.

value of g_1 has been shifted by 90°. This results in a merger time which is about seven percent longer than that shown in figure 1.

The overall behavior of the eccentricity oscillations depicted in figures 1(b) and 2(b) can be understood by examining the evolution in the e_1 vs. g_1 phase space. The initial phase space trajectories for triples with the same total angular momentum as the triples of figures 1 and 2 are depicted in figure 3. Depending on the initial conditions, the evolution in g_1 is either one of libration about $g_1 = 90°$ or 270°, or one of circulation. The quadrupolar fixed point of the Kozai resonance lies inside the smallest libration contour at $\cos g_1 = 0$. The only difference between the triples of figures 1 and 2 is that the former starts off circulating while the latter starts off librating. Figure 4 depicts snapshots of the phase space trajectories during the course of the evolution shown in figures 1 and 2. As shown in figure 4(b), gravitational radiation drives an initially librating inner binary over into a circulating inner binary, thereby causing the minimum eccentricity to drop until it crosses the separatrix. This separatrix crossing corresponds to the momentary zero eccentricity spike just before 10^7 years in figure 2(b). Thereafter, the evolution is very similar to the case where the inner binary starts off circulating: the minimum eccentricity rises until gravitational radiation

becomes so strong that it starts to circularize the orbit.

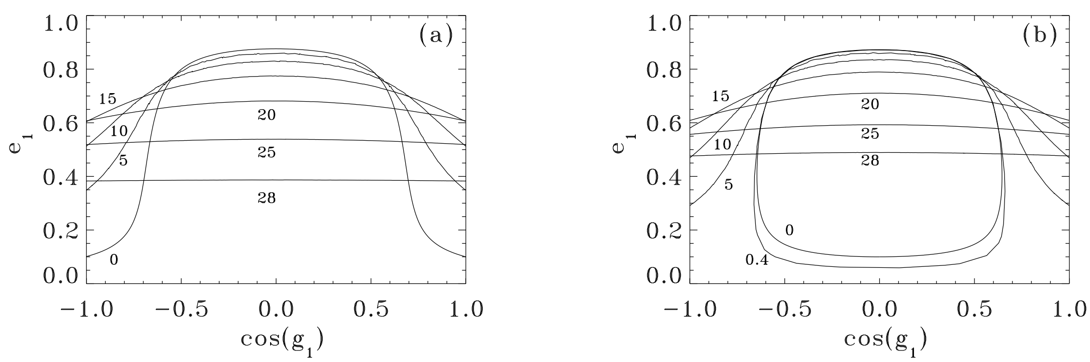

Fig. 4.— Snapshots of the e_1 vs. $\cos g_1$ phase space during the course of the evolution depicted in (a) figure 1 (initially circulating in g_1) and (b) figure 2 (initially librating in g_1). Each curve is labeled with the time in units of 10^7 yr.

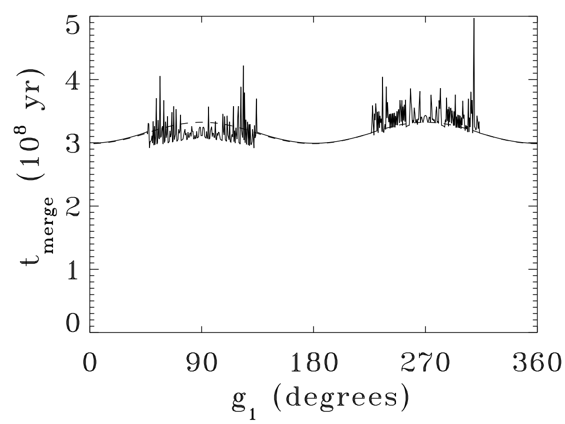

Fig. 5.— Merger time as a function of the initial argument of periastron g_1 of the inner binary, for a triple consisting of black hole masses $m_0 = 2 \times 10^6$ $M_\odot$ and $m_1 = m_2 = 10^6$ $M_\odot$. The initial conditions of the triple are $a_1 = 3.16 \times 10^{-3}$ pc, $a_2/a_1 = 10$, $e_1 = 0.1$, $e_2 = 0.1$, $g_2 = 90°$, and $i = 80°$. The solid line depicts the merger time as calculated with our full equations of motion, while the dashed curve is the time obtained by neglecting the octupole terms.

Figure 5 shows the overall dependence of the merger time for varying initial inner argument of periastron g_1, again for an inner orbit which is initially roughly circular: $e_1 = 0.1$. While the overall effect of g_1 on the merger time is not too great, the exact value of the merger time is extremely sensitive to the initial value of g_1 for triples close to the libration/circulation separatrix, or for those triples which start out librating (g_1 initially around 90° or 270°) and therefore subsequently cross the separatrix. As shown in figure 3, the region near the separatrix is chaotic, and this chaos

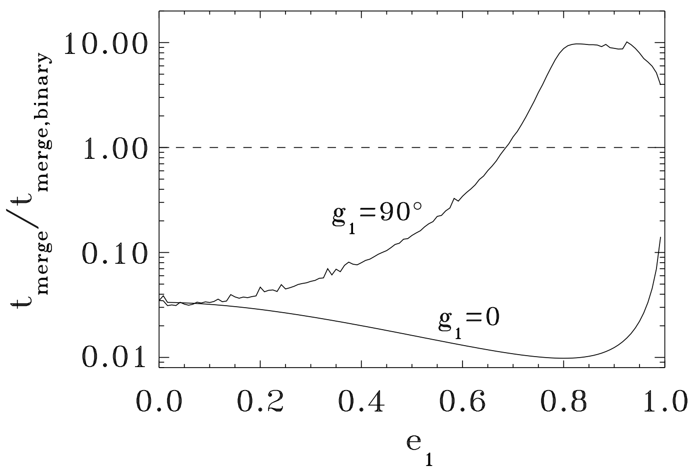

Fig. 6.— Merger time as a function of the initial eccentricity e_1 of the inner binary, for a triple consisting of black hole masses $m_0 = 2 \times 10^6$ $M_\odot$ and $m_1 = m_2 = 10^6$ $M_\odot$. The initial conditions of the triple are $a_1 = 3.16 \times 10^{-3}$ pc, $a_2/a_1 = 10$, $e_2 = 0.1$, $g_2 = 90°$, and $i = 80°$. Two initial values of g_1 are shown: $g_1 = 0$ (initially circulating) and $g_1 = 90°$ (initially librating for low e_1, circulating for high e_1). In contrast to previous figures, we have scaled the merger time with the nominal binary merger time calculated neglecting interactions with the outer black hole, equation (2), which also depends sensitively on e_1. The Kozai mechanism speeds up the merger in all cases where the initial eccentricity is low, and can even speed up the merger at high initial eccentricity in the case of $g_1 = 0$.

arises from the octupole interaction terms. If these terms are neglected, the system then lacks the necessary degrees of freedom to exhibit chaos, and the dependence of the merger time on the initial value of g_1 is much smoother (the dashed line in figure 5). Note that the separatrix is associated with passing through very small values of e_1 [cf. figure 2(b)]. One might therefore worry that our results suffer from numerical problems associated with the $1/e_1$ singularity in the octupole term of equation (12). However, following the suggestion of Ford et al. (2000), we have removed this singularity in our numerical integrations by first transforming the dependent variables from (g_1, e_1, g_2, e_2) to $(e_1 \cos g_1, e_1 \sin g_1, e_2 \cos g_2, e_2 \sin g_2)$. In addition, the evolution of the Hamiltonian, equation (24), is completely smooth through the separatrix crossing, changing only as a result of gravitational radiation losses. We therefore believe the "noise" exhibited in figure 5 is physical, and is caused by chaotic behavior during evolution through the separatrix.

The initial orientation of the inner orbit affects the merger time much more substantially when the inner orbit is more eccentric. Figure 6 depicts the merger time as a function of the initial inner eccentricity e_1 for two values of the initial inner argument of periastron: $g_1 = 0$ and $90°$. The former case corresponds to a circulating inner binary which starts out with an eccentricity which

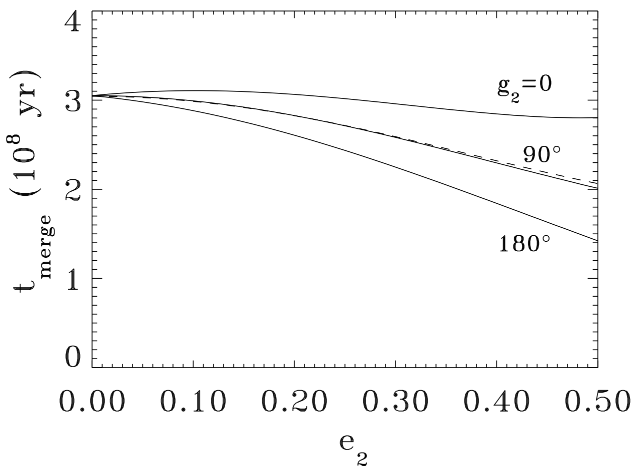

Fig. 7.— Merger time as a function of the initial eccentricity e_2 of the outer binary, for a triple consisting of black hole masses $m_0 = 2 \times 10^6$ M$_\odot$ and $m_1 = m_2 = 10^6$ M$_\odot$. The initial conditions of the triple are $a_1 = 3.16 \times 10^{-3}$ pc, $a_2/a_1 = 10$, $g_1 = 0$, and $i = 80°$. Different solid curves correspond to different choices for the initial argument of periastron g_2 of the outer binary. The dashed line shows the merger time when octupole terms in the evolution equations are neglected, in which case g_2 does not affect the evolution of the inner binary.

is at the minimum in the Kozai oscillation (see figure 3). Hence in this case the Kozai mechanism *always* speeds up the merger compared to the time $t_{\rm merge,binary}$ it would take an isolated binary with the same initial eccentricity to merge. In contrast, inner binaries with initial $g_1 = 90°$ start at the minimum in the Kozai eccentricity oscillation for values of e_1 which are below the Kozai fixed point, and at the maximum for values of e_1 above it. We would therefore expect the Kozai mechanism to reduce the merger time in the former case and increase it in the latter. This expectation is confirmed by the behavior shown in figure 6, provided the binary does not start out too near the Kozai fixed point which occurs at $e_1 \simeq 0.8$ for this set of triples with fixed mutual inclination angles. Near the Kozai fixed point, the merger time is generally increased by the Kozai mechanism, due perhaps to the fact that gravitational radiation drives the eccentricity at the Kozai fixed point down to lower values with time. We therefore conclude that the Kozai mechanism always acts to reduce the merger time of an initially nearly circular binary, but for eccentric inner binaries it will either reduce or increase the merger time, depending on the orientation of the inner binary within the triple system.

The $g_1 = 90°$ curve in figure 6 exhibits the chaos we typically find for inner binaries which start out librating, or circulating near the separatrix. This chaos is also present at the lowest values of eccentricity e_1 in the $g_1 = 0$ curve, as the inner binary then starts out in the chaotic zone shown in figure 3.

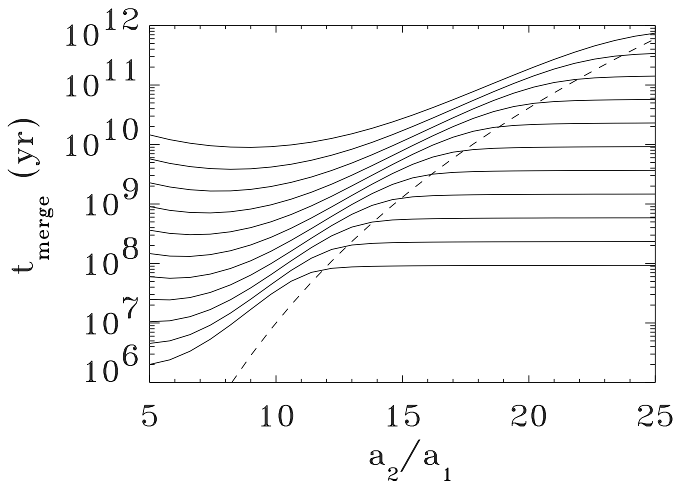

Fig. 8.— Merger time of an inner binary black hole system with masses $m_0 = 2 \times 10^6$ M$_\odot$ and $m_1 = 10^6$ M$_\odot$ in a hierarchical triple with outer black hole mass $m_2 = 10^6$ M$_\odot$, as a function of the initial semimajor axis ratio a_2/a_1 of the triple. The initial conditions of the triple are $e_1 = 0.1$, $e_2 = 0.1$, $g_1 = 0$, $g_2 = 90°$, and $i = 80°$. From bottom to top, the different solid curves show results for different initial semimajor axes of the inner binary, spaced at equal logarithmic intervals: $a_1 = \{1.00, 1.26, 1.58, 2.00, 2.51, 3.16, 3.98, 5.01, 6.31, 7.94, 10.0\} \times 10^{-3}$ pc. The dashed curve separates the region on the left where the Kozai resonance exists (at least initially) from that on the right where general relativistic precession destroys the eccentricity oscillations.

We turn now to the effects of the orientation and eccentricity of the outer binary on the merger time of the inner binary. Equations (12)-(13) and (18)-(19) show that increasing the eccentricity e_2 of the outer binary at fixed semimajor axis a_2 strengthens both the quadrupolar and octupolar interaction terms, with the latter being enhanced over the former. This is of course physically reasonable as the distance of closest approach of the outer black hole with the inner binary is smaller. On the other hand, the outer argument of periastron g_2 only affects the evolution of the inner binary through the octupolar interaction terms. We would therefore expect that increasing e_2 would generally decrease the merger time through the (quadrupolar) Kozai mechanism, but that the initial value of g_2 could modify this significantly at high eccentricity due to the enhanced octupolar effects. These expectations are confirmed by our numerical calculations shown in figure 7. Note, however, that the effects of e_2 and g_2 are not that large, at least for this particular triple. We have therefore chosen to fix their initial values to be $e_2 = 0.1$ and $g_2 = 90°$ for all our other numerical calculations.

Perhaps the most important initial condition parameters affecting the inner binary merger time are the semimajor axes a_1 and a_2 and the mutual inclination of the inner and outer orbits. Figure 8 shows the dependence of the merger time on the semimajor axis ratio a_2/a_1 for triples

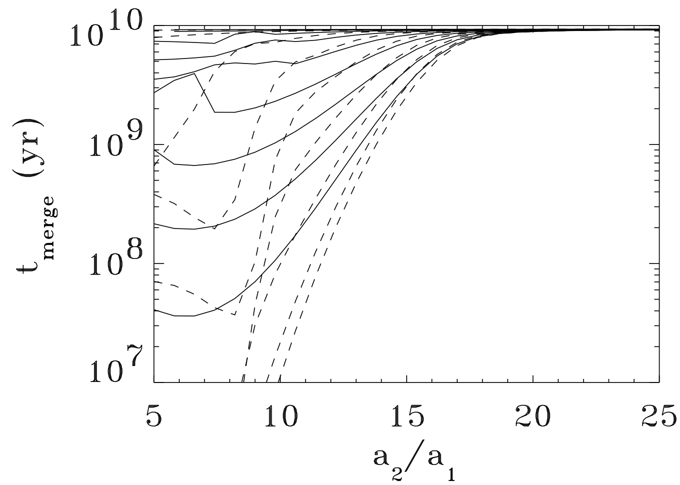

Fig. 9.— Merger time of an inner binary black hole system with masses $m_0 = 2 \times 10^6$ M$_\odot$ and $m_1 = 10^6$ M$_\odot$ in a hierarchical triple with outer black hole mass $m_2 = 10^6$ M$_\odot$, as a function of the initial semimajor axis ratio a_2/a_1 of the triple. The different curves correspond to different initial mutual inclinations of the binary: from bottom to top, $|\cos i|$ ranges from 0.1 to 0.9 in steps of 0.1. (The 0.8 and 0.9 curves lie almost on top of each other.) Prograde outer orbits ($\cos i > 0$) are shown by solid curves, while retrograde outer orbits ($\cos i < 0$) are shown by dashed curves. The other initial conditions of the triple are $a_1 = 3.16 \times 10^{-3}$ pc, $e_1 = 0.1$, $e_2 = 0.1$, $g_1 = 0$, and $g_2 = 90°$. The irregular behavior at intermediate inclinations and low values of a_2/a_1 is a result of chaos.

with fixed initial inclination $i = 80°$ (the same as in all previous figures) and various values of the initial inner semimajor axis a_1. For large initial semimajor axis ratios a_2/a_1 (to the right of the dashed line in figure 8), general relativistic precession destroys the Kozai resonance and the eccentricity of the inner binary is largely unaffected by the outer black hole. The merger time in this case is then the same as that of an isolated binary, and is given by equation (2). On the other hand, if the outer black hole comes sufficiently close (to the left of the dashed line in figure 8), then the Kozai resonance exists and substantial reduction in the merger time occurs as a result of eccentricity oscillations. The equation for the dashed line itself which separates these two regimes comes from using the binary merger time in equation (2) as a proxy for the inner semimajor axis a_1, and employing equation (5):

$$\begin{aligned} t_{\rm merge} &= 1.2 \times 10^6 {\rm yr} \left(\frac{a_2/a_1}{10}\right)^{12} \left(\frac{2m_2}{m_0+m_1}\right)^{-4} \left(\frac{m_0}{10^6 {\rm M}_\odot}\right)^{-1} \left(\frac{m_1}{10^6 {\rm M}_\odot}\right)^{-1} \\ & \left(\frac{m_0+m_1}{2\times 10^6 {\rm M}_\odot}\right)^{3} \frac{(1-e_2^2)^6}{(1-e_1^2)^{5/2}} f(e_1). \end{aligned} \tag{28}$$

Figure 9 shows how the variation of merger time with initial semimajor axis ratio depends on

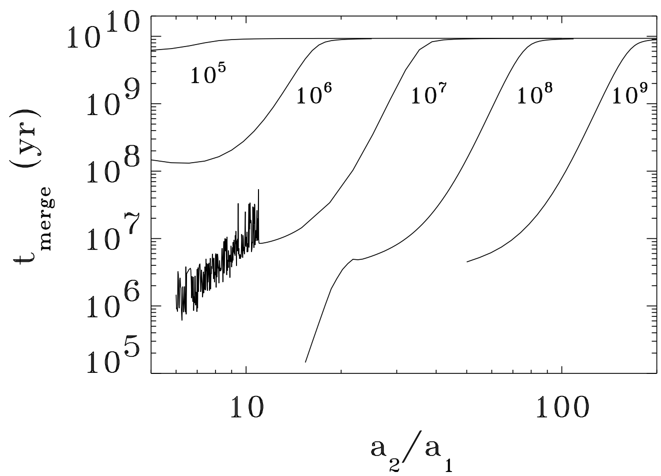

Fig. 10.— Merger time of an inner binary black hole system with masses $m_0 = 2 \times 10^6$ M$_\odot$ and $m_1 = 10^6$ M$_\odot$ in a hierarchical triple, as a function of the initial semimajor axis ratio a_2/a_1 of the triple. The different curves correspond to different outer black hole masses, as labeled in solar masses. The initial conditions of the triple are $a_1 = 3.16 \times 10^{-3}$ pc, $e_1 = 0.1$, $e_2 = 0.1$, $g_1 = 0$, $g_2 = 90°$, and $i = 80°$.

the initial mutual inclination angle of the triple for initial $a_1 = 3.16 \times 10^{-3}$ pc. As expected, significant reduction in the merger time can only occur when the initial inclination is high enough that the Kozai resonance is present: $|\cos i| \lesssim (3/5)^{1/2} \simeq 0.77$. If this criterion is satisfied, then values of a_2/a_1 satisfying equation (5), i.e. $a_2/a_1 < 17.7$ for this particular triple, will exhibit accelerated mergers, with higher inclinations showing the fastest merger times. As Miller & Hamilton (2002) point out for the stellar mass case, retrograde triples ($\cos i < 0$) generally produce faster merger times, at least for nearly equal masses.

4.2. Merger Times for Other Combinations of Masses

As we noted above in section 3, our numerical results for the nearly equal mass triples of the previous subsection can be scaled to all triples with the same mass ratio, $m_0 : m_1 : m_2 = 2 : 1 : 1$, provided the initial semimajor axes are scaled by the same factor as the masses. The merger time then scales by exactly the same factor. We have also investigated triples consisting of substantially unequal masses, and our results may also be scaled to other, similar mass ratio triples in the same manner.

Figure 10 depicts the merger time for the same inner binary as shown in figure 8, but with different outer black hole masses. All cases shown start with an initial inner semimajor axis of

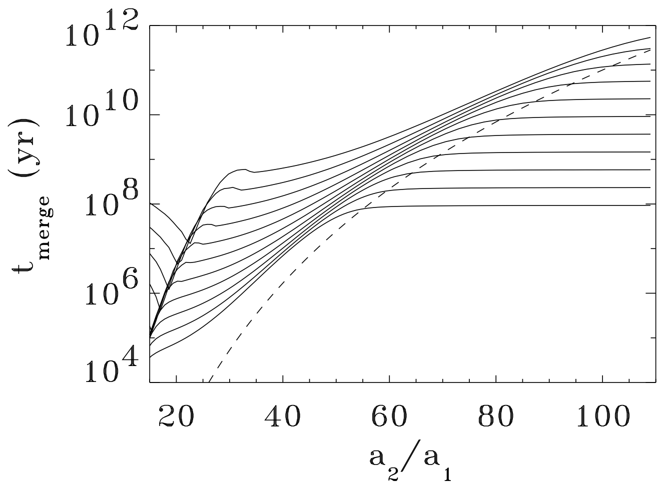

Fig. 11.— Same as figure 8 except for an outer black hole mass of $m_2 = 10^8$ M$_\odot$.

$a_1 = 3.16 \times 10^{-3}$ pc, for which the merger time would be $\simeq 9.3 \times 10^9$ yr if the binary were isolated. Because we do not account for gravitational wave losses associated with the outer binary, all the calculations shown in the figure have a_2/a_1 large enough so that the nominal gravitational wave merger time of the outer binary from equation (2) is at least ten times longer than that of the inner binary.

The 10^5 M$_\odot$ curve shows that an outer black hole with a much smaller mass than that of the inner binary does not have a substantial effect on the merger time of that binary. On the other hand, a larger outer black hole mass exerts a stronger tidal perturbation on the inner binary. This reduces the merger time significantly compared to the nearly equal mass case, and also does it at larger semimajor axis ratios. In agreement with equation (5), the value of a_2/a_1 required for the onset of the Kozai resonance scales with the outer black hole mass as $m_2^{1/3}$. Triples in which the outer black hole mass is much larger than that of the inner binary might arise from the merger of a smaller galaxy containing a stalled binary with a larger galaxy containing a bigger black hole.

Figure 11 presents a more detailed look at the dependence of the merger time on the semimajor axes a_1 and a_2 for the case of an outer black hole mass of $m_2 = 10^8$ M$_\odot$. With the exception of the larger outer black hole mass, the initial triple parameters are identical to those of figure 8. Note the change in scales on the axes: the larger outer black hole mass greatly reduces the merger time at much larger values of a_2/a_1. The dashed line comes once again from equation (28), and separates the region on the left where the Kozai resonance exists from that on the right where relativistic periastron precession destroys the resonance.

Figure 12 shows results for a substantially unequal mass inner binary consisting of 10^8 and 10^6 M$_\odot$ black holes, in a triple with an outer 10^8 M$_\odot$ black hole. We choose initial inner semimajor

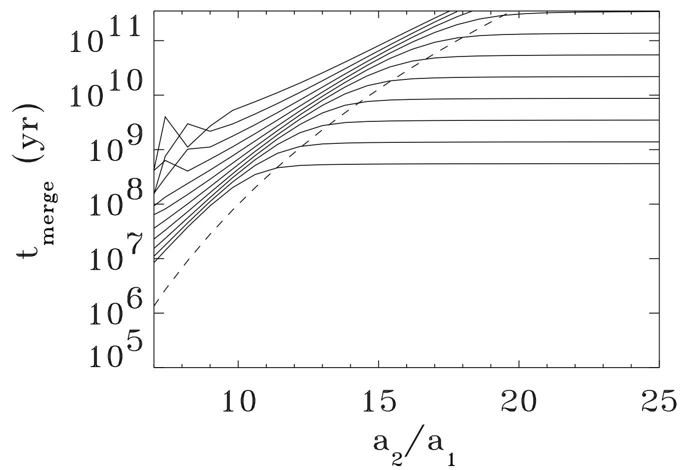

Fig. 12.— Merger time of an inner binary black hole system with masses $m_0 = 10^8$ M$_\odot$ and $m_1 = 10^6$ M$_\odot$ in a hierarchical triple with outer black hole mass $m_2 = 10^8$ M$_\odot$, as a function of the initial semimajor axis ratio a_2/a_1 of the triple. The initial conditions of the triple are $e_1 = 0.1$, $e_2 = 0.1$, $g_1 = 0$, $g_2 = 90°$, and $i = 80°$. From bottom to top, the different solid curves show results for different initial semimajor axes of the inner binary, spaced at equal logarithmic intervals: $a_1 = \{1.00, 1.26, 1.58, 2.00, 2.51, 3.16, 3.98, 5.01, 6.31, 7.94, 10.0\} \times 10^{-2}$ pc. (These values of a_1 are ten times those of figures 8 and 11.) The Kozai resonance exists initially only to the left of the dashed line.

axes a_1 to be ten times larger than those of figures 8 and 11 so that $t_{\rm merge,binary}$ is roughly the same. Very substantial reductions in the merger time occur once again, but for semimajor axis ratios a_2/a_1 that are even smaller than in the roughly equal mass case. Larger outer black hole masses can expand this range of a_2/a_1 (see equation [5]), but would also accelerate the gravitational wave merger of the outer binary, increasing the likelihood of an unstable encounter.

We have only shown results here for prograde orbits in triples with substantially unequal masses. In contrast to the nearly equal mass case, retrograde orbits do not produce merger times which are very different from their prograde counterparts. This asymmetry between prograde and retrograde orbits arises from terms in the equations of motion which depend on odd powers of θ, which are neglected in the usual Kozai approximation (see Appendix A). These terms are also very small for the unequal mass cases we have calculated here.

5. DISCUSSION AND CONCLUSIONS

We have shown that in cases where a bound hierarchical black hole triple forms, substantial reductions in the gravitational wave merger time of the inner binary can take place for reasonably large ranges of initial conditions of the triple. For example, figure 9 shows that for a triple consisting of nearly equal mass black holes, the merger time can be reduced by more than a factor of ten in over fifty percent of all cases, provided the mutual inclination angle is randomly distributed in solid angle. However, we have not addressed the issue of what the distribution of initial conditions is likely to be, as that would entail following the dynamical formation of the triple itself by interactions with the surrounding stars and gas. As we have shown in the previous section, all the initial parameters of the triple affect whether and by how much the merger of the inner binary is altered by the presence of the outer black hole. However, the most important are the mutual inclination angle i and the semimajor axes a_1 and a_2. There is very little effect on the merger time of the inner binary unless the semimajor axis ratio a_2/a_1 is small enough that general relativistic precession does not destroy the Kozai resonance (equation [5]).

An issue related to the uncertainty in initial conditions is the fact that we have not considered the evolution of the semimajor axis of the outer black hole due either to interactions with surrounding material or to gravitational radiation. The latter is not significant for the calculations shown here, as we have always chosen a_2/a_1 and e_2 to be such that the gravitational wave evolution time scale of the outer black hole is much longer than the merger time scale of the inner binary. Note that e_2 evolves purely due to octupolar interactions (equation [16]), and generally does not undergo the dramatic changes that e_1 exhibits. Hence we do not expect dramatic reductions in the *outer* binary merger time scale due to interactions with the inner binary.

However, that still leaves open the issue of whether the inner binary will merge before interactions with surrounding material cause the outer black hole to come sufficiently close that an unstable three body encounter occurs. It would be interesting to follow the evolution of the triple by including the time dependence of a_2 starting at large enough values that the Kozai resonance is destroyed by general relativistic periastron precession.

In addition to the Kozai mechanism, the presence of a third black hole may affect the merger time scale of the inner binary in other ways. In particular, being bound in a triple forces the inner binary to "wander" through space, albeit in a regular orbit rather than a stochastic trajectory. This may enhance the binary's interactions with surrounding stars beyond what it would have had if the outer black hole were not present, and therefore the hardening rate might be increased.

Eccentricity oscillations in the inner binary may be induced by other sources of tidal gravitational fields besides that of an orbiting outer black hole, e.g. nearby matter inhomogeneities or an aspherical distribution of surrounding stars and gas. In order to beat general relativistic periastron precession, equation (A6) implies that the quadrupole moment ρ_{2m} of these exterior matter

distributions must satisfy

$$\rho_{2m} \sim \frac{m_2}{4\pi a_2^3} > 2\times 10^6 \mathrm{M}_\odot\ \mathrm{pc}^{-3} \left(\frac{m_0+m_1}{2\times 10^6\mathrm{M}_\odot}\right)^2 \left(\frac{a_1}{10^{-2}\mathrm{pc}}\right)^{-4} (1-e_1^2)^{-3/2}. \tag{29}$$

Note that this critical quadrupole moment has a strong scaling with the binary semimajor axis a_1, and it may be possible to achieve given the actual stellar mass densities observed in galactic nuclei (e.g. Faber et al. 1997).

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A. APPENDIX: ANALYTIC ESTIMATES NEGLECTING OCTUPOLE TERMS

In this appendix we present analytic calculations which are useful in understanding how general relativistic precession modifies the Kozai mechanism. We neglect the octupole terms in the hierarchical triple evolution equations (11)-(17) throughout this section. The only parameter of the outer binary which then changes is g_2, and this evolution does not affect the inner binary.

Neglecting gravitational radiation, the evolution of the inner binary is determined solely by equations (12) and (13):

$$\begin{aligned} \frac{dg_1}{dt} &= 6C_2\left\{\frac{1}{G_1}[4\theta^2+(5\cos 2g_1-1)(1-e_1^2-\theta^2)]+\frac{\theta}{G_2}[2+e_1^2(3-5\cos 2g_1)]\right\} \\ &\quad +\frac{3}{c^2 a_1(1-e_1^2)}\left[\frac{G(m_0+m_1)}{a_1}\right]^{3/2}, \end{aligned} \tag{A1}$$

and

$$\frac{de_1}{dt} = 30C_2\frac{e_1(1-e_1^2)}{G_1}(1-\theta^2)\sin 2g_1. \tag{A2}$$

Kozai (1962) has provided an approximate analytic solution to these quadrupole evolution equations when all general relativistic effects are neglected. The approximation neglects the term multiplied by θ/G_2 in equation (A1) which is smaller than the first term by $G_1/G_2 \sim (a_1/a_2)^{1/2}$ for comparable masses. Also, equation (22) implies that

$$2G_1G_2\theta + G_1^2 = H^2 - G_2^2, \tag{A3}$$

which is a constant when octupole terms and gravitational radiation are neglected. Again taking $G_2 >> G_1$, this gives an approximate integral of motion

$$\Theta \equiv (1 - e_1^2)\theta^2 \simeq \frac{m_0 + m_1}{G a_1 m_0^2 m_1^2} \left(\frac{H^2 - G_2^2}{2G_2} \right)^2 . \tag{A4}$$

For $\Theta < 3/5$, Kozai's solutions exhibit two classes of dynamical behavior in the evolution of the argument of perihelion g_1: libration about $g_1 = 90°$ or $270°$, and circulation. There then exists a critical inclination angle for which the librating solutions degenerate to a fixed point in the e_1 vs. g_1 phase space: $\cos^2 i_{\rm crit} = 3(1 - e_1^2)/5$. For $\Theta > 3/5$, the resonance vanishes and only circulating solutions exist. In this case the amplitude of eccentricity oscillations is quite small, so the existence of the resonance can be used as a necessary criterion for determining when large amplitude eccentricity oscillations become possible. Because $\Theta < 3/5$ implies that $\cos^2 i < 3/5/(1 - e_1^2)$, large amplitude oscillations are only possible for high inclinations: $39° \simeq \cos^{-1}(3/5)^{1/2} < i < 180° - \cos^{-1}(3/5)^{1/2} \simeq 141°$.

General relativistic precession is itself circulation in g_1, so general relativity would be expected to reduce the parameter space where librations exist. Equation (A2) implies that the fixed point, if it exists, still occurs at $g_1 = 90°$ or $270°$. If we continue to neglect the term multiplied by θ/G_2 in equation (A1), then setting $dg_1/dt = 0$ gives

$$\cos^2 i_{\rm crit} = \frac{3}{5}(1 - e_1^2) - \frac{4G(m_0 + m_1)^2 a_2^3 (1 - e_2^2)^{3/2}}{5c^2 m_2 a_1^4 (1 - e_1^2)^{1/2}}. \tag{A5}$$

As expected, general relativistic precession decreases $\cos^2 i_{\rm crit}$, so that the fixed point can only exist if the inclination angle is pushed higher. This can only be achieved so long as the right hand side of equation (A5) remains positive, implying that

$$\frac{a_2^3}{a_1^3} < \frac{3c^2 m_2 a_1 (1 - e_1^2)^{3/2}}{4G(m_0 + m_1)^2 (1 - e_2^2)^{3/2}}, \tag{A6}$$

which gives equation (5).

Equation (A5) can be rearranged to give an equation for the eccentricity $e_{1,0}$ at the fixed point for triples with specified total angular momentum H, and therefore specified Kozai constant Θ within the Kozai approximation,

$$1 - e_{1,0}^2 = \left[\frac{5\Theta}{3} + \frac{4G(m_0 + m_1)^2 a_2^3 (1 - e_2^2)^{3/2} (1 - e_{1,0}^2)^{1/2}}{3c^2 m_2 a_1^4} \right]^{1/2} . \tag{A7}$$

Solving this equation for $e_{1,0}$ requires solving a quartic, but written in this way it is again obvious that general relativistic precession lowers the value of $e_{1,0}$ for triples of given angular momentum, thereby increasing the circulating portion of phase space at the expense of libration.

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